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Riemannian Geometry and Multilinear Tensors with Vector Fields on Manifolds

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Abstract-In the paper some aspects of Riemannian manifolds, pseudo-Riemannian manifolds, Lorentz manifolds, Riemannian metrics, affine connections, parallel transport, curvature tensors, torsion tensors, killing vector fields, conformal killing vector fields are focused. The purpose of this paper is to develop the theory of manifolds equipped with Riemannian metric. I have developed some theorems on torsion and Riemannian curvature tensors using affine connection. A Theorem 1.20 named "Fundamental Theorem of Pseudo-Riemannian Geometry" has been established on Riemannian geometry using tensors with metric. The main tools used in the theorem of pseudo Riemannian are tensors fields defined on a Riemannian manifold.

Keywords: Riemannian manifolds, pseudo-Riemannian manifolds, Lorentz manifolds, Riemannian metrics, affine connections, parallel transport, curvature tensors, torsion tensors, killing vector fields, conformal killing vector fields.

I. Introduction

Riemannian manifold is a pair (M, g) consisting of smooth manifold M and Riemannian metric g. A manifold may carry a further structure if it is endowed with a metric tensor, which is a natural generation of the inner product between two vectors in $\mathbb{R}^n$ to an arbitrary manifold. Riemannian metrics, affine connections, parallel transport, curvature tensors, torsion tensors, killing vector fields and conformal killing vector fields play important role to develop the theorem of Riemannian manifolds.

II. Riemannian manifolds

A manifold is a topological space which locally looks like $\mathbb{R}^n$. Calculus on a manifold is assured by the the existence of smooth coordinate systems. Indeed, Riemannian manifold is the generalization of Riemannian metric with smooth manifold.

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Definition 1.01 M is an *n-dimensional differentiable manifold* if

(a) M is a topological space,

(b) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$,

(c) $\{U_i\}$ is a family of open sets which covers M, that is, $\bigcup_i U_i = M$.

(d) φ_i is a homeomorphism from U_i onto an open subset U_i' of $\mathbb{R}^n$.

(e) Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_i \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

Example 1.02 The Euclidean space $\mathbb{R}^m$ is the most trivial example, where a single chart covers the whole space and φ may be the identity map.

Definition 1.03 Let $\varphi_i : U_i \to \mathbb{R}^n$ be a homeomorphism from an open subset U_i into $\mathbb{R}^n$. Then the pair (U_i, φ_i)) is called a *chart*.

Definition 1.04 [1] Let M be a differentiable manifold. A *Riemannian metric* g on M is a type $(0,2)$ tensor field on M which satisfies the following axioms at each point $p \in M$

(a) $\mathrm{g}_p(U,V) = \mathrm{g}_p(V,U)$

(b) $\mathrm{g}_p(U,U) \geq 0$ where the equality holds only when $U = 0$.

Here $U,V \in T_pM$ and $\mathrm{g}_p = \mathrm{g}|_p$.In short g_p is a symmetric positive definite bilinear form and T_pM is a tangent space of manifold M at a point p.

Definition 1.05 Let M be a differentiable manifold. A Riemannian metric g on M is a *pseudo-Riemannian metric* if it satisfies the conditions (i) and (ii) and if $\mathrm{g}_p(U,V) = 0$ for any $U \in T_pM$, then $V = 0$.

Definition 1.06 If g is Riemannian matric, all the eigenvalues are strictly positive and if g is pseudo-Riemannian some eigenvalues are negative. If there are i positive and j negative eigenvalues, then the pair (i,j) is called the *index of metric*. If $i = j$, the metric is called a *Lorentz metric*.

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Definition 1.07 Let (M, g) is Lorentzian. The elements of $T_p M$ are divided into three classes as follows

(a) $\mathrm{g}(U,U) > 0 \to U$ is *spacelike*,

(b) $\mathrm{g}(U,U) = 0 \to U$ is *lightlike*,

(c) $\mathrm{g}(U,U) < 0 \to U$ is *timelike*.

Definition 1.08 [2] If a smooth manifold M admits a Riemannian metric g, the pair (M, g) is called a *Riemannian manifold*. If g is a pseudo-Riemannian metric, then (M, g) is said to be a *pseudo-Riemannian manifold*. If g Lorentzian, (M, g) is called a *Lorentz manifold*.

Example 1.09 An m-dimensional Euclidian space $(\mathbb{R}^m, \delta)$ is Riemannian manifold and an m-dimensional Minkowski space $(\mathbb{R}^m, \eta)$ is a Lorentz manifold.

III. Affine connection and covariant derivative

A vector X is a directional derivative acting on $f \in \mathcal{F}(M)$ as $X : f \to X(f)$. However, there is no directional derivative acting on a tensor field of type (p, q) which arises naturally from the differentiable structure of M. What we need is an extra structure called the connection, which how tensor are transported along a curve.

Definition 1.10 [3] Let M be a smooth n-dimensional manifold, $\mathcal{F}(M)$ be the set of smooth functions and $\mathfrak{X}(M)$ be the vector space of smooth vector fields. An affine connection on M is a map which is denoted by ∇ and defined by

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$(X, Y) \mapsto \nabla_X Y$$

Such that

(a) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

(b) $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

(c) $\nabla_X (f\, Y) = X(f)\, Y + f\, \nabla_X Y$

(d) $\nabla_{f X} Y = f\, \nabla_X Y$,

$\forall f \in \mathcal{F}(M)$ and $X, Y \in \mathfrak{X}(M)$.

Definition 1.11 Let (U, φ) be a coordinate chart on a manifold M with a coordinates $(x^1, x^2, \dots, x^n)$. The functions $\Gamma^k_{ij}(x)$ are called *coordinate symbols* of the affine connection ∇. Here $\Gamma^k_{ij}(x)$ is a n^3 function, where $i, j = \overline{1, n}$.

Definition 1.12 The vector field $\nabla_X f$ is often called *covariant derivative* of vector field $f \in \mathcal{F}(M)$ along the vector field X. It is to define the covariant derivative of f by the ordinary directional derivative,

$$\nabla_X f = X(f).$$

For any f, Y, it can be defined as follows

$$\nabla_X (f\, Y) = (\nabla_X f) Y + f\, \nabla_X Y.$$

Definition 1.13 Let T_1 and T_2 be two tensor fields. Then the *covariant derivative* ∇_X along the field X is defined as follows

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2).$$

IV. Parallel Transport

Given a curve in a manifold M, we may define the parallel transport of a vector along the curve. In geometry, parallel transport is a way of transporting geometrical data along smooth curves in a manifold. If the manifold is equipped with an affine connection, then this connection allows one to transport vectors of the manifold along curves so that they stay parallel with respect to the connection.

Definition 1.14 Let M be a smooth n-dimensional manifold equipped with an affine connection ∇. Let $\gamma: (a, b) \to M$ be a smooth curve. A vector field on $\gamma(t)$, $X(t)$ is called *parallel transport* if the following equation is satisfied

$$\nabla_{\dot{\gamma}(t)} X(t) = 0, \quad \forall\ t \in (a, b).$$

Here $\dot{\gamma}(t)$ is the tangent vector to $\gamma(t)$ at the point t.

Theorem 1.15 [4] Let $\gamma: (a, b) \to M$ be a smooth curve on a manifold M. For each $t_0 \in (a, b)$ and for each $X_0 \in T_{\gamma(t_0)} M$, prove that there exist a unique vector field $X(t)$ on $\gamma(t)$ such that

(a) $X(t)$ is parallel,

(a) $X(t_0) = X_0$.

Proof. Let (U, φ) be a coordinate chart on a manifold M at t_0 with coordinates $(x^1, x^2, \dots, x^n)$. Then

(a) A smooth curve $\gamma : (a, b) \to M$ is given by a set on n smooth functions

$$\left.\begin{array}{l} x^1 = x^1(t) \\ x^2 = x^2(t) \\ \quad \vdots \\ \quad \vdots \\ x^n = x^n(t) \end{array}\right\} \Rightarrow x^i = x^i(t),$$

where $t \in (a, b), i = 1, 2, \dots, n$

(a) The vector $X(t)$ is given by

$$X(t) = \sum_{i=1}^n X^i(t) \frac{\partial}{\partial x^i} \quad \text{for}$$

$$= X^1(t) \frac{\partial}{\partial x^1} + X^2(t) \frac{\partial}{\partial x^2} + \dots + X^n(t) \frac{\partial}{\partial x^n}.$$

Then $\dot{\gamma}(t)$ is given by

$$\dot{\gamma}(t) = \sum_{i=1}^n \frac{dx^i}{dt} \frac{\partial}{\partial x^i}$$

$$= \frac{dx^1}{dt}\cdot\frac{\partial}{\partial x^1} + \frac{dx^2}{dt}\cdot\frac{\partial}{\partial x^2} + \cdots + \frac{dx^n}{dt}\cdot\frac{\partial}{\partial x^n}.$$

Now we have,

$$\nabla_{\dot{\gamma}(t)} X(t) = \nabla_{\sum_{i=1}^n \frac{dx^i}{dt} e_i} \sum_{j=1}^n X^j\ e_j$$

$$= \sum_{i=1}^n \frac{dx^i}{dt}\ \nabla_{e_i}\left(\sum_{j=1}^n X^j\ e_j\right)$$

$$= \sum_{i=1}^n \frac{dx^i}{dt}\left[\ \sum_{j=1}^n e_i\ (X^j)\ e_j + \sum_{j=1}^n X^j\ \nabla_{e_i} e_j\ \right]$$

$$= \sum_{i=1}^n \sum_{j=1}^n \left[\ (\frac{dx^i}{dt}\ e_i X^j) e_j + \frac{dx^i}{dt}\ X^j\ \nabla_{e_i} e_j\ \right]$$

$$= \sum_{i=1}^n \sum_{j=1}^n \left[\ (\frac{dx^i}{dt}\frac{\partial X^j}{\partial x^i})\frac{\partial}{\partial x^j} + \frac{dx^i}{dt}\ X^j \sum_{k=1}^n \Gamma_{ij}^k(x)\ \frac{\partial}{\partial x^k}\right]$$

$$= \sum_{j=1}^n \frac{dX^j}{dt}\cdot\frac{\partial}{\partial x^j} + \sum_{i,j,k=1}^n \frac{dx^i}{dt}\ X^j\ \Gamma_{ij}^k(x)\ \frac{\partial}{\partial x^k}$$

$$\Rightarrow \nabla_{\dot{\gamma}(t)} X(t) = \sum_{k=1}^n \left[\frac{dX^k}{dt} + \sum_{i,j=1}^n \frac{dx^i}{dt}\ X^j\ \Gamma_{ij}^k(\mathrm{x})\right]\frac{\partial}{\partial x^k}.$$

Thus the equation for the parallel transport is

$$\left.\begin{array}{l} \frac{dX^k}{dt} + \sum_{i,j=1}^n \frac{dx^i}{dt}\ X^j\ \Gamma_{\mathrm{ij}}^{\mathrm{k}}(x) = 0 \\ X^k(t_0) = X^k{}_0 \quad (initial\ condition) \end{array}\right\}$$

This is a system of n-equations for n-unknown functions $X^k(t)$ with n-initial conditions. A theorem from the theory of differential equations says that the solution exists and which is unique. This completes the proof of this theorem.

V. Torsion tensor and Riemann curvature tensor

In the mathematical field of differential geometry, the Riemann curvature tensor, or Riemannian–Christoffel tensor is the most standard way to express curvature of Riemannian manifolds. It associates a tensor to each point of a Riemannian manifold that measures the extent to which the metric tensor is not locally isometric to a Euclidean space.

Definition 1.16 [5] Let M be a smooth n-dimensional manifold, $\mathcal{F}(M)$ be the set of smooth functions and $\mathfrak{X}(M)$ be the vector space of smooth vector fields. A *tensor A* of rank $(1,p)$ on M is a multi-linear map

$$A\ :\ \mathfrak{X}(M)\times\mathfrak{X}(M)\times \ldots \times \mathfrak{X}(M) \ \to \mathcal{F}(M)$$

which satisfies

$$A(f\ x_1, x_2, \ldots, x_p) = A(\ x_1, f x_2, \ldots\ , x_p) = \cdots$$

$$= A(\ x_1, x_2, \ldots, f x_p)$$

$$= f A(\ x_1, x_2, \ldots, x_p)$$

for any function $f \in \mathcal{F}(M)$ and $x_1,\ x_2,\ \ldots,\ x_p \in \mathfrak{X}(M)$.

Definition 1.17 A *torsion* T^∇ of an affine connection ∇, is a map

$$T^\nabla:\ \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$$

$$(X,Y)\mapsto T^\nabla(X,Y)$$

where $T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

Theorem 1.18 For all affine connection ∇, its torson T^∇ is a tensor of rank $(1,2)$.

Proof. We need to prove that
$T^\nabla(f\,X,Y) = T^\nabla(X, f\,Y) = f\,T^\nabla(X,Y),\ \forall f \in \mathcal{F}(M)$ and $X, Y \in \mathfrak{X}(M)$.

By the definition of torsion, we get

$$T^\nabla(f\,X,Y) = \nabla_{f\,X} Y - \nabla_Y f\,X - [\,f\,X,Y\,]$$

$$= f\,\nabla_X Y - Y(f)X - f\,\nabla_Y\ X - [\,f\,X,Y\,] \quad \ldots\ \ldots\ \ldots\ (1.01)$$

Now take any g $\in \mathcal{F}(M)$, then

$$[\,f\,X,Y\,]\mathrm{g} = f\,X(\,Y(\mathrm{g})) - Y(f\,X(\mathrm{g}))$$

$$= f\,XY(\mathrm{g}) - Y(f).X(\mathrm{g}) - f\,YX(\mathrm{g})$$

$$= f\,(\,XY(\mathrm{g}) - YX(\mathrm{g})) - Y(f).X(\mathrm{g})$$

$$= f\,[X,Y]\mathrm{g} - Y(f).X(\mathrm{g})$$

$$\therefore [\,f\,X,Y\,] = f\,[X,Y] - Y(f)\,X.$$

Therefore, equation (1.01) becomes

$$T^\nabla(f\,X,Y) = f\,\nabla_X Y - Y(f)X - f\,\nabla_Y\ X - f\,[X,Y] + Y(f)\,X$$

$$= f\,(\nabla_X Y - \nabla_Y\ X - [X,Y])$$

$$= f\,T^\nabla(X,Y).$$

Again,

$$T^\nabla(X, fY) = -T^\nabla(fY, X)$$

$$[\because T^\nabla(X,Y) = -T^\nabla(Y,X)]$$

$$= -f\ T^\nabla(Y,X) \quad \text{[as previous part]}$$

$$= f\,T^\nabla(X,Y).$$

Therefore, T^∇ is a tensor of rank $(1,2)$.

Hence completes the proof.

Definition 1.19 The *curvature tensor* R^∇, of an affine connection ∇, is a map

$R^{\nabla}: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$

$$(X, Y, Z) \mapsto R^{\nabla}(X, Y, Z)$$

where $R^{\nabla}(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Theorem 1.20 [6] For all affine connection ∇, its curvature R^{∇} is a tensor of rank $(1,3)$.

Proof: We need to prove that

$$R^{\nabla}(fX, Y, Z) = R^{\nabla}(X, fY, Z)$$
$$= R^{\nabla}(X, Y, fZ)$$
$$= f R^{\nabla}(X,Y,Z);\ \forall f \in \mathcal{F}(M) \text{ and } X,\ Y,\ Z \in \mathfrak{X}(M).$$

By the definition of curvature tensor, we get

$$R^{\nabla}(fX, Y, Z)$$
$$= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} - \nabla_{[fX,Y]} Z$$
$$= f \nabla_X \nabla_Y Z - \nabla_Y (f \nabla_X Z) - \nabla_{f[X,Y]-Y(f)X} Z$$
$$= f \nabla_X \nabla_Y Z - Y(f)\ \nabla_X Z - f \nabla_Y\ \nabla_X Z - \nabla_{f[X,Y]} Z + \nabla_{Y(f)X} Z$$
$$= f \nabla_X \nabla_Y Z - Y(f)\ \nabla_X Z - f \nabla_Y\ \nabla_X Z - f \nabla_{[X,Y]} Z + Y(f) \nabla_X Z$$
$$= f\ (\nabla_X \nabla_Y Z - \nabla_Y\ \nabla_X Z - \nabla_{[X,Y]} Z)$$
$$= f R^{\nabla}(X, Y, Z).$$

Again, note that

$$R^{\nabla}(X,Y,Z) = -R^{\nabla}(Y,X,Z)$$

Then

$$R^{\nabla}(X, fY, Z) = -R^{\nabla}(fY, X, Z)$$
$$= -f R^{\nabla}(Y,X,Z) \quad \text{[as previous part]}$$
$$= f R^{\nabla}(X, Y, Z).$$

Also,

$$R^{\nabla}(X, Y, fZ)$$
$$= \nabla_X \nabla_Y f Z - \nabla_Y \nabla_X f Z - \nabla_{[X,Y]} f Z$$
$$= \nabla_X (Y(f)Z + f \nabla_Y Z - \nabla_Y (X(f).Z + f \nabla_X Z)$$
$$-[X,Y](f).Z\ f \nabla_{[X,Y]} Z$$
$$= \nabla_X (Y(f)Z) + \nabla_X (f \nabla_Y Z) - \nabla_Y (X(f).Z)$$
$$-\nabla_Y (X(f).Z) - \nabla_Y (f \nabla_X Z) - [X,Y](f).Z$$
$$-f \nabla_{[X,Y]} Z$$
$$= \nabla_X (Y(f)Z) + Y(f) \nabla_X Z + X(f) \nabla_Y Z$$
$$+ f \nabla_X \nabla_Y Z - Y(X(f))Z - X(f) \nabla_Y Z$$
$$- Y(f) \nabla_X Z - f\ \nabla_Y \nabla_X Z - [X,Y]f.Z$$
$$-f \nabla_{[X,Y]} Z$$
$$= (XY(f) - YX(f))Z + f(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$$
$$-\nabla_{[X,Y]} Z) - [X,Y]f.Z$$
$$= [X,Y]f.Z + f R^{\nabla}(X, Y, Z) - [X,Y]f.Z$$
$$= f R^{\nabla}(X, Y, Z).$$

Therefore, R^{∇} is a tensor of rank $(1,3)$.
Hence completes the proof.

VI. Levi-Civita connections

Let M be a smooth n-dimensional manifold, $\mathcal{F}(M)$ be the set of smooth functions, g be a smooth metric, $\mathfrak{X}(M)$ be the vector space of smooth vector fields and ∇ be an affine connection on M. Then the covariant derivative on g with respect to ∇ is a multilinear map,

$$\nabla g : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathcal{F}(M)$$
$$(Z, X, Y) \mapsto \nabla_Z g(X, Y)$$

where $\nabla_Z g(X,Y) = Z g(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$

Definition 1.21 [7] Let M be a smooth manifold equipped with a smooth manifold metric g. There is a unique affine connection ∇ on g such that

(a) ∇ is torsion free.

(b) $\nabla g = 0$ This unique connection is called *Levi-Civita connection*.

Theorem 1.22 (The fundamental theorem of pseudo-Riemannian geometry) On a pseudo-Riemannian manifold (M, g), there exists a unique symmetric connection (Levi-Civita connection) which is compatible with the metric g.

Proof: Let γ be a tangent vector to an arbitrary curve along which the vectors are parallel transported. Then we have,

$$0 = \nabla_{\gamma}[g(X,Y)] = \gamma^{i}[(\nabla_i g)(X,Y)] + g(\nabla_i X, Y)$$
$$+ g(X, \nabla_i Y)$$
$$= \gamma^{i} X^{j} Y^{k} (\nabla_i g)_{jk}).$$

where we have noted that $\nabla_i X = \nabla_i Y = 0$. Since this true for any curves and vectors, we must have

$$(\nabla_i g)_{jk} = 0.$$

For metric tensor we know,

$$(\nabla_{\ell} g)_{jk} = \frac{\partial}{\partial x^{\ell}} g_{jk} - \Gamma^{i}_{\ell j} g_{ik} - \Gamma^{i}_{\ell k} g_{ij}.$$

Now from the above, we can write as follows

$$\frac{\partial}{\partial x^{\ell}} g_{jk} - \Gamma^{i}_{\ell j} g_{ik} - \Gamma^{i}_{\ell k} g_{ij} = 0 \quad \dots \dots \dots (1.02)$$

Now using cyclic permutations of (ℓ, j, k), we have

$$\frac{\partial}{\partial x^{j}} g_{k\ell} - \Gamma^{i}_{jk} g_{i\ell} - \Gamma^{i}_{j\ell} g_{ik} = 0 \dots \dots \dots (1.03)$$

$$\frac{\partial}{\partial x^{k}} g_{\ell j} - \Gamma^{i}_{k\ell} g_{ij} - \Gamma^{i}_{kj} g_{i\ell} = 0 \dots \dots \dots (1.04)$$

The combination $-(1.02) + (1.03) + (1.04)$ yields

$$-\frac{\partial}{\partial x^{\ell}} g_{jk} + \frac{\partial}{\partial x^{j}} g_{k\ell} + \frac{\partial}{\partial x^{k}} g_{\ell j} + T^{i}_{\ell j} g_{ik}$$
$$+ T^{i}_{\ell k} g_{ij} - 2\Gamma^{i}_{(j\,k)} g_{i\ell} = 0 \dots \dots \dots (1.05)$$

where $T^{i}_{\ell j} = 2\Gamma^{i}_{[\ell j]} = \Gamma^{i}_{\ell j} - \Gamma^{i}_{j \ell}$ and $\Gamma^{i}_{(j\,k)} = \frac{1}{2}\left(\Gamma^{i}_{kj} + \Gamma^{i}_{j\,k}\right)$. The tensor $T^{i}_{\ell j}$ is anti-symmetric with respect to the lower indices $T^{i}_{\ell j} = -T^{i}{}_{j\ell}$.

By solving equation (1.05) for $\Gamma^{i}_{(j\,k)}$, we have

$$\Gamma^{i}_{(j\,k)} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix} + \frac{1}{2}\left(T^{i}_{k\;j} + T^{i}_{j\,k}\right) \dots \dots \dots (1.06)$$

where $\begin{Bmatrix} i \\ j\,k \end{Bmatrix}$ are the christoffel symbols defined by

$$\begin{Bmatrix} i \\ j\,k \end{Bmatrix} = \frac{1}{2}\, g^{i\,l}\left(\frac{\partial g_{k\,l}}{\partial x^{j}} + \frac{\partial g_{j\,l}}{\partial x^{k}} - \frac{\partial g_{j\,k}}{\partial x^{\ell}}\right) \dots \dots \dots (1.07)$$

Finally, the connection coefficient Γ is given by

$$\Gamma^{i}_{j\,k} = \Gamma^{i}_{(j\,k)} + \Gamma^{i}_{[j\,k]}$$
$$= \begin{Bmatrix} i \\ j\,k \end{Bmatrix} + \frac{1}{2}\left(T^{i}_{k\;\;j} + T^{i}_{j\;\;k} + T^{i}{}_{j\,k}\right) \dots \dots \dots (1.08)$$

The second term of the last expression of (1.08) is called contorsion, denoted by $K^{i}{}_{jk}$:

$$K^{i}{}_{jk} = \frac{1}{2}\left(T^{i}_{k\;\;j} + T^{i}_{j\;\;k} + T^{i}{}_{j\,k}\right)$$

So,

$$\Gamma^{i}_{j\,k} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix} + K^{\;i}_{j\,k}$$

Now, $\Gamma^{\;i}_{(j\,k)} = \Gamma^{\;i}_{j\,k} + T^{\;i}_{j\,k}$ is the another connection coefficient if T is a tensor field of type (1,2). Now we choose

$T^{\;i}_{j\,k} = -K^{\;i}_{j\,k}$ so that

$$\Gamma^{\;i}_{j\,k} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix}$$
$$= \frac{1}{2}\, g^{i\,l}\left(\frac{\partial g_{lk}}{\partial x^{j}} + \frac{\partial g_{j\,l}}{\partial x^{k}} - \frac{\partial g_{j\,k}}{\partial x^{l}}\right).$$

By construction, this is symmetric and certainly unique given a metric.

Example 1.23 Let metric on $\mathbb{R}^2$ in polar coordinates is $g = dr \otimes dr + r^2 d\phi \otimes d\phi$. The non-vanishing components of the Levi-Civita connection coefficients are $\Gamma^{\;\phi}_{r\phi} = \Gamma^{\;\phi}_{r\lambda} = r^{-1}$ and $\Gamma^{\;r}_{\phi\phi} = -r$.

Example 1.24 The induce map on S^2 is $g = d\theta \otimes d\theta + sin^2\theta\, d\phi \otimes d\phi$. the non-vanishing components of the Levi-Civita connections are

$$\Gamma^{\;\theta}_{\phi\phi} = -cos\theta\, sin\theta\,;\; \Gamma^{\;\phi}_{\theta\phi} = \Gamma^{\;\phi}_{\phi\theta} = cot\theta.$$

VII. Killing Vector Fields

A Killing vector field is a vector field on a Riemannian manifold that preserves the metric. Killing fields are the infinitesimal generators of isometrics, that is, flows generated by Killing fields are continuous isometrics of the manifold.

Definition 1.25 [9] A vector field X is a *Killing vector field* if the Lie derivative with respect to X of the metric g vanishe

$$\mathcal{L}_x g = 0.$$

In terms of the Levi-Civita connection, this is

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

for all vectors Y and Z. In local coordinates, this amounts to the Killing equation

$$\nabla_i X_j + \nabla_j X_i = 0.$$

This condition is expressed in covariant form. Therefore it is sufficient to establish it in a preferred coordinate system in order to have it hold in all coordinate systems.

Examples 1.26 The vector field on a circle that points clockwise and has the same length at each point is a Killing vector field, since moving each point on the circle along this vector field simply rotates the circle.

VIII. Conformal Killing Vector Fields

The term of the conformal Killing vector field is an extension of the term of the Killing vector field. Conformal Killing vector fields scale the Metric around a smooth function, while Killing vector fields do not scale the Metric. The conformal Killing vectors are the infinitesimal generators of conformal transformations.

Definition 1.27 [10] Let (M, g) be a Riemannian manifold and $X \in \mathfrak{X}(M)$. Then the vector field X is a *conformal Killing vector field*, if an infinitesimal displacement given by $\varepsilon\, X$ generates a conformal transformation.

Example 1.28 Let x^k be the coordinates of $(\mathbb{R}^m, \delta)$. The vector

$$D = x^k \frac{\partial}{\partial x^k}$$

is a conformal killing vector.

IX. Conclusion

The fundamental theorem of pseudo-Riemannian geometry is established using tensors on a manifold M. In this theorem, I have

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used metric connection ∇ which is the natural generalization of the connection defined in the classical geometry of surfaces. The covariantly constant metric g_{ij} and vectors fields X and Y, which are parallel transported along any curve are used in this theorem. In this paper, I have tried to set different types of examples and the proof of various theorems in elaborate way so that it can be helpful for further analysis.

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